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The Second Noether Theorem in the formalism of jet-bundles: Symmetries and degeneration

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Abstract

By means of the jet-bundle formalism, the Second Noether Theorem is formulated for a general first-order Lagrangian field theory with infinitesimal local symmetries. These symmetries are implemented by a linear differential operator acting between the sections of a vector bundle and vector fields on the configuration bundle. The problem of the degeneration of the Lagrangian system is examined from a covariant and an instantaneous (i.e. space+time split) viewpoint. It is shown that in the instantaneous approach the presence of infinitesimal local symmetries leads to degeneration of the theory. Vertical local symmetries are shown to imply degeneration also in the covariant formalism. These results can be extended to higher-order Lagrangians as well.

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1. Introduction

In classical Field Theories with gauge invariance, the Second Noether Theorem is the main tool to show non-regularity of the theory and to give conditions on constructing Lagrangians which possess a given gauge symmetry [28,17]. The term "gauge invariance", however, is rather ambiguous, because it has been extended from the electromagnetic and Yang–Mills theories to the general case of field theories – e.g. General Relativity, String Theory – exhibiting symmetries depending on arbitrary functions on the space–time manifold and on

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their derivatives up to some finite order. For this reason in our paper we will adopt instead the expression "local symmetry".

Attempts to give a general definition of local symmetry and a formulation of the Second Noether Theorem can be found in [16], and with a more intrinsic setting in [3].

Here we assume that the fields of the theory are sections of a general fiber bundle (E, π, M) on a space–time manifold M, and Lagrangians are defined on a jet bundle of sections. In this we were inspired by [9], where a formulation of the calculus of variations, a general notion of symmetry and a version of the Noether Theorem are given. In this work, a symmetry of a Lagrangian system is formulated as a wide concept concerning the invariance of the action functional under transformations of the space of sections upon which this functional is defined.

We focus our attention on first-order Lagrangians; in the last section we will give an account of the generalization to the higher-order case of the results of the preceding sections. In order to formulate the Noether Theorem we will adopt here the same attitude as in [30] (see also [29] for a definition of symmetry and the Noether Theorem for higher-order Lagrangian systems) and we will consider only those symmetries represented by automorphisms of the fiber bundle (E, π, M); hence, infinitesimal symmetries are represented by vector fields on E projectable on M. About the notion of local symmetries, we confine our discussion to *infinitesimal* local symmetries.

We assume that the infinite parameters of the symmetries are sections of a vector bundle (A, p, M) and a *differential operator* $\mathcal{P}$ *along* π is given which transforms these sections into projectable vector fields on E; these fields are taken to be symmetries of the Lagrangian system. These assumptions are justified, since in many cases the gauge transformation group admits the structure of an infinite-dimensional Lie group, and its Lie algebra consists of sections of a suitable bundle (see e.g. [5] for a review on this topic). These observations were our starting point for the development of such a formalism. However, as a first approximation, we disregard the "Lie-algebraic" aspects of the space of sections of (A, p, M) and the properties which the operator $\mathcal{P}$ must satisfy in this respect. We believe that this point of view may prove to be valuable not only with respect to the question of degeneration, to which we focus our attention, but also for the effective construction of the momentum map associated with the action of a gauge group, even if for this subject, the algebraic aspects are of crucial importance. In this context, we propose this work as a first step along the determination of the geometric aspects of a gauge system.

In Section 3 we recall the notion of symmetry of a Lagrangian system and the First Noether Theorem. In the case of Lagrangians with infinitesimal local symmetries, the *generalized Bianchi identities* are given by means of a suitable non-linear operator $\mathcal{D}$ (Section 4). Then we come to the problem of non-regularity of the Lagrangian function possessing a local symmetry.

There are different notions of *regularity*, depending on which Hamiltonian formulation is assumed. Several covariant Hamiltonian formulations of a general Lagrangian field theory were proposed: we quote to the interested reader the extensive introduction in [10]; this work also gives an exhaustive list of references, and an exposition of the multisymplectic structure, which seems to us to be the most natural between the possible covariant Hamiltonianizations of field theories. We will not deal directly with it since we will only consider the Lagrangian aspects.

Anyway – at least in the case of first-order field theories – the starting point for a covariant Hamiltonian formulation is the De Donder–Cartan equation, and regularity means, at the very end, the equivalence between the Euler–Lagrange and the De Donder–Cartan equations, as it is discussed e.g. in [9].

In the "space+time split" approach to an Hamiltonian formulation perhaps – the most usual in the common practice – the starting point is the construction of an infinite-dimensional manifold $\mathcal{Q}$ and of an instantaneous Lagrangian function on its tangent bundle $T\mathcal{Q}$; therefore one treats field theories in a way that resembles point-particle theories. Even in this case one can give a notion of regularity without any reference to the underlying Hamiltonian formalism, by saying that the Lagrangian system is regular if the Lagrange two-form appearing in the equations of motion (5.3) is non-degenerate (i.e. it is a symplectic form on $T\mathcal{Q}$).

In Section 5 we perform a naive splitting of space–time inducing a splitting on the jet bundle of sections of (E, π, M), and we construct the manifold of initial data which, adding some hypotheses, can be identified with a tangent manifold. We define here the instantaneous Lagrangian, the energy function and the Lagrange two-form as in [11]. Maybe a more refined treatment can be given, and the splitting of the jet bundle might be realized in more generality (perhaps by means of a connection on E), but leading to analogous results with respect to the degeneration.

As a consequence of the Second Noether Theorem, simply considering that the symbol of the differential operator $\mathcal{D}$ must vanish identically, we prove non-regularity of a Lagrangian with an infinitesimal local symmetry in the space+time split approach: it is an expected result, at least in field theory for Lagrangians on flat space–time or, more trivially, in a local frame of reference.

In Section 6 we are able to prove the non-regularity of the covariant Hamiltonian formalism for Lagrangians with a vertical local symmetry. We observe that this is the case for the Yang–Mills-type Lagrangians, i.e. one of the most relevant examples of gauge theory, which are degenerate both in the covariant and in the instantaneous formalism. This double degeneration, however, is not characteristic of this kind of Lagrangians: we show the example of a Lagrangian with a gauge-fixing term, and we give arguments showing the impossibility of defining vertical local symmetries for it.

In Section 7 we give a brief account on the generalization to the higher-order Lagrangians of the interplay between symmetries and degeneration. Even if in this case there is no uniquely defined Cartan form, it is shown that a local symmetry implies degeneration in the instantaneous formalism, and a vertical local symmetry implies degneration of the covariant Legendre transformations associated to the Poincaré–Cartan forms.

Future developments might involve the Lie-algebraic aspects of these local symmetries, relevant for the construction of the momentum map, which in turn appears to be an important tool in understanding the BRST formalism and constraints generated by a gauge group. Moreover, it might be very interesting to develop a "covariant constraint theory" and a covariant algorithm *à la Dirac*. The main point here would be the link between local symmetries and constraints, compared with the similar relation existing between gauge invariance and first-class constraints in the instantaneous formalism. Other developments might involve the extension of the concept of symmetry. In this respect, some extensions of the Second Noether Theorem have been proposed in particle mechanics: see e.g. [19,20]. It might be interesting to analyze how one should enlarge the concept of symmetry so as to produce similar extensions in the instantaneous formalism.

2. Notations

Let (E, π, M) be a (general) fiber bundle, where E and M are Hausdorff and paracompact finite-dimensional smooth manifolds. M will also be assumed to be orientable. We will sometimes use the shorthand notation E_M instead of (E, π, M). By (TM, τ_M, M) we will denote the tangent bundle and by VE the vertical bundle of E, i.e. the kernel of the tangent projection $T\pi$. If E is the total space of different bundles, we will use e.g. the notation VE_M, to avoid confusion. The space of smooth sections of (E, π, M) will be denoted by $\Gamma(\pi)$ or $\Gamma(E_M)$.

The key objects in the present paper are the jet-bundles over E and their geometry, for which the main reference, here and along all the work, is [25]. We will let (J^k, π_k, M) be the fiber bundle of the k-jets of sections of (E, π, M), where $J^0 = E$ and $\pi^0 = \pi$; J will stand for J^1. We denote by $\pi_{k,l}$, (for $k > l$), the natural projection $J^k \longrightarrow J^l$. We will also denote by $j^k s$ the jet-prolongation of the section s, and $j^k s$ itself will be called a *holonomic section* of J^k. Local coordinates on M will be denoted by x^μ, $\mu = 0, \dots, m$, the fibered coordinates on E by (x^μ, y^a), with $a = 1, \dots, r$, and finally those on J^k by $\left(x^\mu, y^a, y^a_N\right)$, where N is a multi-index of length at most k: $|N| \leq k$.

It is well known that $\left(J^{k+1}, \pi_{k+1,k}, J^k\right)$ is structured as an affine fiber bundle, and that its associated vector bundle is $VE \otimes_{J^k} \bigvee^k T^*M$. This is a shorthand notation for the tensor product of the vector bundles $\pi^*_{k,0} VE$ and $\pi^*_k \bigvee^k T^*M$. Then, specializing to the case $k = 0$, VJ_E can be shown to be diffeomorphic to $VE \otimes_J T^*M$ in a natural way by means of a map $i : VE \otimes_J T^*M \longrightarrow VJ_E$ (see [30]), which is locally given by $i : \partial/\partial y^a \otimes \mathrm{d}x^\mu \longmapsto \partial/\partial y^a_\mu$.

Regarding $\left(J^k, \pi_k, M\right)$ as a general fiber bundle, we have the well known exact sequence $0 \longrightarrow VJ^k_M \longrightarrow TJ^k \longrightarrow \pi^*_k TM \longrightarrow 0$ and by pullback the following one in which the space $\mathcal{T}_k$ and $\mathcal{V}_k$ are also defined:

$$0 \longrightarrow \mathcal{V}_k = (\pi_{k+1,k})^* \left(VJ^k_M\right) \longrightarrow \mathcal{T}_k = (\pi_{k+1,k})^* \left(TJ^k\right) \longrightarrow \pi^*_{k+1}(TM).$$

This sequence does possess a canonical splitting. One can indeed define the following injective vector-bundle morphism called the *holonomic lift*:

$$\lambda_k : \pi^*_{k+1}(TM) \longrightarrow \mathcal{T}_k, \qquad \lambda_k(e_{k+1}, u) = \left(e_{k+1}, T_x j^k s(u)\right),$$

where $s \in e_{k+1} \in J^{k+1}$ and $x = \pi_{k+1}(e_{k+1}) = \tau_M(u)$. Defining $\mathcal{H}_k = \mathrm{Im}(\lambda_k)$ one has the canonical decomposition $\mathcal{T}_k = \mathcal{H}_k \oplus \mathcal{V}_k$. The space $\Gamma\left(\mathcal{T}_{k\,J^{k+1}}\right)$ of sections of $\mathcal{T}_k$ turns out to be a $C^\infty\left(J^{k+1}\right)$-module.

A local basis for $\Gamma\left(\mathcal{T}_{k\,J^{k+1}}\right)$ is given by $\partial/\partial x^\mu, \partial/\partial y^a, \dots, \partial/\partial y^a_N, |N| \le k$.

The sections of the sub-bundle $\mathcal{H}_{k\,J^{k+1}}$ are called *total derivatives*; a local basis is given by

$$\left(\frac{\mathrm{d}}{\mathrm{d}x^\mu}\right), \quad \mu = 0, \dots, m,$$

where

$$\frac{\mathrm{d}}{\mathrm{d}x^\mu} = \frac{\partial}{\partial x^\mu} + \sum_{|N| \le k} y^a_{N+\mu} \frac{\partial}{\partial y^a_N},$$

and the sum $N + \mu$ is defined by $N + \mu = (n_0, \dots, n_\mu + 1, \dots, n_m)$.

The vector bundle $\mathcal{T}^*_k = \left(\pi_{k+1,k}\right)^* T^* J^k$ is the dual vector bundle of $\mathcal{T}_k$ and has the analogous splitting $\mathcal{T}^*_k = \mathcal{H}^*_k \oplus \mathcal{V}^*_k$.

Clearly, one can consider the bundle $\mathcal{T}^*_k$ naturally included into $T^* J^{k+1}$ defining $(e_{k+1}, \alpha)(X) = \alpha(e_k)\left(T_{e_{k+1}} \pi_{k+1,k}(X)\right)$, $\forall X \in T_{e_{k+1}} J^{k+1}, \forall (e_{k+1}, \alpha) \in \mathcal{T}^*_k$, where $e_k = \pi_{k+1,k}(e_{k+1})$.

With a construction which parallels the one on the tangent bundle of a manifold, one can define the vector bundles

$$\bigwedge^p \mathcal{T}^*_k = \left(\pi_{k+1,k}\right)^* \bigwedge^p T^* J^k, \qquad \bigwedge \mathcal{T}^*_k = \left(\pi_{k+1,k}\right)^* \bigwedge T^* J^k.$$

The decomposition $\mathcal{T}^*_k = \mathcal{H}^*_k \oplus \mathcal{V}^*_k$ induces the analogous one $\bigwedge \mathcal{T}^*_k = \bigwedge \mathcal{H}^*_k \otimes \bigwedge \mathcal{V}^*_k$ (where it is meant that each fiber $\left(\bigwedge \mathcal{T}^*_k\right)_{e_{k+1}}$ is the tensor product of the two graded algebras $\bigwedge \left(\mathcal{H}^*_k\right)_{e_{k+1}}$ and $\bigwedge \left(\mathcal{V}^*_k\right)_{e_{k+1}}$).

We will use the symbol Φ_k for the $C^\infty\left(J^{k+1}\right)$-module of sections of $\bigwedge \mathcal{T}^*_k$. Let $\Phi_{k(q)}$ be the space of sections of $\bigwedge^q \mathcal{T}^*_k$ and $\Phi_k{}^{(r,s)}$ be the space of sections of $\bigwedge^r \mathcal{H}_k{}^* \otimes_{J^{k+1}} \bigwedge^s \mathcal{V}_k{}^*$; then $\Phi_{k(q)} = \sum_{r+s=q} \Phi_k{}^{(r,s)}$ and $\Phi_k = \sum_q \Phi_{k(q)} = \sum_{r,s} \Phi_k^{(r,s)}$. As a local basis for the 1-forms on $\mathcal{T}_k$, adapted to the decomposition into "horizontal + vertical" spaces one can take

$$\left\{\mathrm{d}x^\mu\right\}_{\mu=0\dots m} \cup \left\{\mathrm{d}_v y^a_N \equiv \mathrm{d}y^a_N - y^a_{N+\mu}\mathrm{d}x^\mu\right\}_{|N| \le k}.$$

As for the terminology, one will say that the elements of $\Phi_k^{(r,s)}$ are *r-times horizontal* and *s-times vertical* (or contact) forms. We stress the fact that, as usual for pull-back bundles, elements of $\Phi_k^{(r,s)}$ can be considered as maps $J^{k+1} \longrightarrow \bigwedge^r T^*M \otimes_{J^{k+1}} \bigwedge^s T^* J^k$. Moreover, from the canonical inclusion $\mathcal{T}^*_k \subset T^* J^{k+1}$ one can think of Φ_k as included in $A\left(J^{k+1}\right)$, the algebra of forms on J^{k+1}.

We recall the definitions of the horizontal and the vertical differential $\mathrm{d}_{h_k}, \mathrm{d}_{v_k} : A\left(J^k\right) \to A\left(J^{k+1}\right)$ given by the following relations: for $f \in C^\infty(J^k)$,

$$\mathrm{d}_{h_k} f = \frac{\mathrm{d}f}{\mathrm{d}x^\mu}\mathrm{d}x^\mu, \quad \text{where } \frac{\mathrm{d}f}{\mathrm{d}x^\mu} = \frac{\partial f}{\partial x^\mu} + \sum_{|N| \le k} \frac{\partial f}{\partial y^a_N} y^a_{N+\mu}$$

is the total derivative and

$$\mathrm{d}_{v_k} f = \sum_{|N|\le k} \frac{\partial f}{\partial y^a_N} \mathrm{d}_v y^a_N, \qquad \mathrm{d}_{h_k} \mathrm{d}x^\mu = \mathrm{d}_{v_k} \mathrm{d}x^\mu = 0,$$

$$\mathrm{d}_{h_k} \mathrm{d}y^a_N = \mathrm{d}x^\mu \wedge \mathrm{d}y^a_{N+\mu}, \qquad \mathrm{d}_{v_k} \mathrm{d}y^a_N = \mathrm{d}y^a_{N+\mu} \wedge \mathrm{d}x^\mu.$$

Actually, these are the actions of the objects just defined:

$$\mathrm{d}_{h_k} : \Phi^{(r,s)}_k \longrightarrow \Phi^{(r+1,s)}_{k+1}, \qquad \mathrm{d}_{v_k} : \Phi^{(r,s)}_k \longrightarrow \Phi^{(r,s+1)}_{k+1}$$

(for an intrinsic definition of d_{h_k} and d_{v_k} see [25]).

Important properties satisfied by the vertical and horizontal differentials are $\mathrm{d}_{v_k} + \mathrm{d}_{h_k} = (\pi_{k+1,k})^* \mathrm{d}$, $\mathrm{d}_{v_{k+1}} \cdot \mathrm{d}_{v_k} = \mathrm{d}_{h_{k+1}} \cdot \mathrm{d}_{h_k} = \mathrm{d}_{v_{k+1}} \cdot \mathrm{d}_{h_k} + \mathrm{d}_{h_{k+1}} \cdot \mathrm{d}_{v_k} = 0$ and $\forall s \in \Gamma(\pi)$, $(j^k s)^* \mathrm{d} = (j^{k+1} s)^* \mathrm{d}_{h_k}$, $(j^k s)^* \mathrm{d}_{v_k} = 0$. By means of these maps Tulczyjew defined in [31] two coboundary operators d_h, d_v acting on the algebra of differential forms over the bundle J^∞. Accordingly, we will drop the index k from our notation.

If Y is a vector field on E projectable on M, its jet-extension on J^r will be denoted by $Y^{(r)}$. We recall the following local expressions:

$$Y^{(1)} = X^\mu \frac{\partial}{\partial x^\mu} + Y^a \frac{\partial}{\partial y^a} + \left(\frac{\mathrm{d}Y^a}{\mathrm{d}x^\mu} - y^a_\nu \frac{\partial X^\nu}{\partial x^\mu} \right) \frac{\partial}{\partial y^a_\mu},$$

$$\begin{aligned} Y^{(2)} = {} & X^\mu \frac{\partial}{\partial x^\mu} + Y^a \frac{\partial}{\partial y^a} + Y^{(1)a}_\mu \frac{\partial}{\partial y^a_\mu} \\ & + \left(\frac{\mathrm{d}}{\mathrm{d}x^\mu} \frac{\mathrm{d}Y^a}{\mathrm{d}x^\nu} - y^a_{\mu\sigma} \frac{\partial X^\sigma}{\partial x^\nu} - y^a_{\nu\sigma} \frac{\partial X^\sigma}{\partial x^\mu} - y^a_\sigma \frac{\partial^2 X^\sigma}{\partial x^\mu \partial x^\nu} \right) \frac{\partial}{\partial y^a_{\mu\nu}}. \end{aligned}$$

3. Lagrangian dynamics and symmetries

Here and in the following three sections we restrict to first-order Lagrangians for which it is simpler to give the notion of degenerate Lagrangian. Some remarks for an analogous discussion in the case of higher-order Lagrangian are given in the last section. We will define *first-order Lagrangian system* the pair (L, Ω), where $L : J \longrightarrow \mathbb{R}$ is a smooth function (a *Lagrangian*) defined over the first jet-bundle of a fiber bundle (E, π, M), and Ω is a volume form on M. We will eventually use the same notation for the pullback of the form Ω on J^k, $k \ge 1$.

Let us define

$$\lambda(L) : J \longrightarrow V^* J_E, \qquad \lambda(L)(e) = \mathrm{d}_{J_E} L(e) = \mathrm{d}_e L_y,$$

where L_y is the restriction of L to the fiber J_y with $\pi_{1,0}(e) = y$.

Upon use of the transposed of the isomorphism i, we can identify $V^* J_E$ with $V^* E \otimes_J TM$, and projecting down on the base E, we can think of $\lambda(L)$ as a map $J \longrightarrow V^* E \otimes_E TM$, which in local coordinates is given by

$$\lambda(L) : \left(x^\mu, y^a, y^a_\mu\right) \longmapsto \left(x^\mu, y^a, \frac{\partial L}{\partial y^a_\mu}\right). \tag{3.1}$$

Using the vertical projection $v : TJ \longrightarrow TE, v(\xi_j) = T\pi_{1,0}\,\xi_j - Tjs\,\xi_x$ where $\xi_x = T\pi_1\,\xi_j$ and $s \in j(x)$ and composing $\lambda(L)$ with the transposed map of $v \otimes \mathrm{id}$, one gets a vector-valued 1-form $A(\mathrm{d}L)$ on J along the projection π_1 in the sense of Frölicher–Nijenhuis [7], and by inclusion with the totally horizontal m-form Ω one obtains (see [25])

$$\mathbf{Leg}_L(\Omega) = \mathbf{i}_{A(\mathrm{d}L)}\Omega$$

as a form in $\Phi_0^{(m,1)}$. If in local coordinates one has $\Omega = \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^m$, and one defines

$$\Omega_\mu = (-)^\mu \mathrm{d}x^0 \wedge \cdots \wedge \widehat{\mathrm{d}x^\mu} \wedge \cdots \wedge \mathrm{d}x^m,$$

where the hat denotes omission, then one can write that

$$\mathbf{Leg}_L(\Omega) = \frac{\partial L}{\partial y^a_\mu}\mathrm{d}_v y^a \wedge \Omega_\mu.$$

Finally, one can express the Euler–Lagrange form $\mathbf{E}_L(\Omega) \in \Phi_1^{(m+1,1)}$ as $\mathbf{E}_L(\Omega) = \mathrm{d}_v L \wedge \Omega + \mathrm{d}_h\left(\mathbf{Leg}_L(\Omega)\right)$ and the Cartan form $\mathbf{C}_L(\Omega) \in \Phi_1$ as $\mathbf{C}_L(\Omega) = \mathbf{Leg}_L(\Omega) + L\Omega$. Their local expressions are

$$\mathbf{E}_L(\Omega) = \left[\frac{\partial L}{\partial y^a} - \frac{\mathrm{d}}{\mathrm{d}x^\mu}\frac{\partial L}{\partial y^a_\mu}\right]\mathrm{d}_v y^a \wedge \Omega, \qquad \mathbf{C}_L(\Omega) = L\Omega + \frac{\partial L}{\partial y^a_\mu}\,\mathrm{d}_v y^a \wedge \Omega_\mu.$$

For a Lagrangian system (L, Ω), the *action integral* relative to the open and relatively compact set $U \subset M$ is the functional on the space $\Gamma_U(\pi)$ of sections of (E, π, M) defined on a neighbourhood of the closure $\overline{U}$ by

$$I_U[s] = \int_U (js)^*(L\Omega).$$

We will say that a bundle-isomorphism $(g, \hat{g})$ of (E, π, M) is a *symmetry* of the Lagrangian system if it is such that for any U and any local section one has

$$I_U[s] = I_{\hat{g}(U)}[g \cdot s \cdot \hat{g}^{-1}].$$

An *infinitesimal symmetry* is a complete vector field Y on E, which is projectable on a vector field X on M, such that the one-parameter group of automorphisms $(g_t, \hat{g}_t)$ induced by (Y, X) is a symmetry for any $t \in \mathbb{R}$. If $s_t = g_t \cdot s \cdot \hat{g}_t^{-1}$, then one can say that for any U and any section $s \in \Gamma_U(\pi)$

$$\int_U \hat{g}_t^*(js_t)^*(L\Omega) = \int_U (js)^*(L\Omega), \quad \forall s, \forall t.$$

This relation may be rewritten in the form

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{t=0} I_{\hat{g}_t(U)}[s_t] &= \int\limits_U \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{t=0} \hat{g}_t^* \,(js_t)^* \,(L\Omega) \\ &= \int\limits_U \mathbf{L}_X \left(js^* \,(L\Omega)\right) + \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{t=0} \left((js_t)^* \,(L\Omega)\right) \\ &= \int\limits_U \mathrm{d}\big(X\rfloor \left(js^* \,(L\Omega)\right)\big) + \Big(Y^{(1)} \cdot js - Tjs \cdot X\Big)\rfloor \,(\mathrm{d}L \wedge \Omega) \\ &= \int\limits_U \mathrm{d}\big(X\rfloor \left(js^* \,(L\Omega)\right)\big) + j^2 s^* \Big(Y^{(2)}\rfloor \mathrm{d}_v L \wedge \Omega\Big) = 0, \end{aligned} \tag{3.2}$$

where $\mathbf{L}_X$ is the Lie derivative along X, and $X\rfloor\alpha$ is the inclusion of the vector field X and the form α. Considering now the definition of $\mathbf{E}_L(\Omega)$, Eq. (3.2) becomes

$$0 = \int\limits_U \mathrm{d}\Big(X\rfloor js^* \,(L\Omega) + js^* Y^{(1)}\rfloor \mathbf{Leg}_L(\Omega)\Big) + j^2 s^* \Big(Y^{(2)}\rfloor \mathbf{E}_L(\Omega)\Big). \tag{3.3}$$

If we consider the case in which s is a solution over U of the Euler–Lagrange equations, then we obtain the vanishing of an exact form:

$$0 = \mathrm{d}\Big(X\rfloor \left(js^* \,(L\Omega)\right) + js^* \Big(Y^{(1)}\rfloor \mathbf{Leg}_L(\Omega)\Big) - \alpha\Big),$$

where α is an m-form on M. This is essentially the statement of the First Noether Theorem.

4. The Second Noether Theorem

The Second Noether Theorem is usually stated for Lagrangians which admit local symmetries: roughly speaking, the action integral is invariant under the action of a group (denoted $G_{\infty r}$) whose parameters depend on r arbitrary functions and on their derivatives up to order k. Groups of this kind have been studied; they have been given the structure of infinite-dimensional Lie groups, and their algebras (the infinitesimal symmetries) are Lie algebras of sections of suitable bundles (see [5]). We intend to formalize the described framework considering only infinitesimal local symmetries, and disregarding the Lie-algebraic aspects.

Let (A, p, M) be a vector bundle of rank q; $\Gamma(p)$ and $\Gamma(\tau_E)$ will stand respectively for the Fréchét spaces of smooth sections of p, and of smooth vector fields on E. We will say that a continuous linear operator $\mathcal{P} : \Gamma(p) \longrightarrow \Gamma(\tau_E)$ is *local along* π if $\mathcal{P}\sigma_{|\pi^{-1}(U)} = 0$ when $\sigma_{|U} = 0$. We will assume that the vector field $\mathcal{P}\sigma$ is projectable on the field X_σ on M, $\forall\sigma$. Then one can show that its local expression is given by

$$\mathcal{P}\sigma\,(x, y) = \sum_{|N| \le k} D^N \sigma^i(x) \left[e^\mu_{Ni}(x) \frac{\partial}{\partial x^\mu} + f^a_{Ni}(x, y) \frac{\partial}{\partial y^a} \right],$$

where $\sigma(x) = \sigma^i(x)e_i(x)$, with $\{e_i\}$ a local basis of $\Gamma(p)$. (For a proof, one can follow the same line as in [6, p.289], taking into account the projectability on M.) We will say that $\mathcal{P}$ is a differential operator along π of order k taking values in projectable vector fields on E.

Given $\mathcal{P}$, one can easily define new operators as follows: for any $s \in \Gamma(\pi)$, let

$$\mathcal{P}_s : \Gamma(p) \longrightarrow \Gamma\left(s^*(TE)\right), \qquad \mathcal{P}_s\sigma(x) = \mathcal{P}\sigma\left(s(x)\right).$$

We will also be interested in the jet-extended versions of these operators, namely

$$\mathcal{P}_s^{(1)} : \Gamma(p) \longrightarrow \Gamma\left(js^*(TJ)\right), \qquad \mathcal{P}_s^{(1)}\sigma = (\mathcal{P}_s\sigma)^{(1)}.$$

An analogous definition holds for the second-jet extension $\mathcal{P}_s^{(2)}$.

The main hypothesis in order to establish the Second Noether Theorem is that $\mathcal{P}\sigma$ is an infinitesimal symmetry of the Lagrangian system, $\forall\sigma \in \Gamma_c(p)$, where $\Gamma_c(p)$ stands for the space of sections of (A, p, M) with compact support. We will say in this case that the Lagrangian system has an *infinitesimal local symmetry*. From (3.3) we have that for any section $s \in \Gamma_U(\pi)$, and for any $\sigma \in \Gamma_c(p)$ whose support lies in U, the following equation holds:

$$\begin{aligned} 0 &= \int_U \mathrm{d}\left(X_\sigma \rfloor js^*\left((L\Omega) + \mathcal{P}_s^{(1)}\sigma\rfloor\mathbf{Leg}_L(\Omega)\right)\right) + \mathcal{P}_s^{(2)}\sigma\rfloor\mathbf{E}_L(\Omega)\left(j^2s\right) \\ &= \int_U \mathcal{P}_s^{(2)}\sigma\rfloor\mathbf{E}_L(\Omega)\left(j^2s\right). \end{aligned} \tag{4.1}$$

Consider now, for any $s \in \Gamma(\pi)$, the action of the transposed operator (see Appendix A)

$${}^{\mathrm{t}}\mathcal{P}_s{}^{(2)} : \Gamma\left(j^2s^*\left(T^*J^2\right)\otimes\bigwedge^m T^*M\right) \longrightarrow \Gamma\left(p^*\otimes\bigwedge^m T^*M\right),$$

where $p^* : A^* \longrightarrow M$ is the dual bundle of p. Then Eq. (4.1) becomes, for any $\sigma \in \Gamma_c(\pi)$ with $\mathrm{supp}(\sigma) \subset U$, and for any $s \in \Gamma_U(\pi)$

$$\begin{aligned} 0 &= \int_U \left\langle\sigma, {}^{\mathrm{t}}\mathcal{P}_s{}^{(2)}\cdot\mathbf{E}_L(\Omega)\left(j^2s\right)\right\rangle \\ &=: \int_U \langle\sigma, \mathcal{D}(s)\rangle. \end{aligned} \tag{4.2}$$

In this equation we have also defined the *non-linear* differential operator, of order $k+2$, $\mathcal{D} : \Gamma(\pi) \longrightarrow \Gamma(p^*\otimes\bigwedge^m T^*M)$, $\mathcal{D}(s) = {}^{\mathrm{t}}\mathcal{P}_s^{(2)}\cdot\mathbf{E}_L(\Omega)\cdot j^2s$, whose explicit form reads

$$\mathcal{D}(s) = \sum_{|N|\le k}(-)^{|N|}D^N\left[\left(\left(\frac{\partial L}{\partial y^a} - \frac{\mathrm{d}}{\mathrm{d}x^\mu}\frac{\partial L}{\partial y^a_\mu}\right)\left(f^a_{N\,i} - y^a_\rho e^\rho_{N\,i}\right)\right)\left(j^2s\right)\right]\epsilon^i\otimes\Omega,$$

where $\{\epsilon^i\}$ is the local basis of $\Gamma(p^*)$ dual to $\{e_i\}$, $i = 1, \ldots, q$.

By (4.2), $\mathcal{D}$ vanishes on all local sections of (E, π, M), i.e. one has the *generalized Bianchi identities*

$$\mathcal{D}(s) = 0, \quad \forall s \in \Gamma(\pi). \tag{4.3}$$

This can be considered as a formulation of the Second Noether Theorem.

Second Noether Theorem. *Let $\mathcal{P} : \Gamma(p) \longrightarrow \Gamma(\tau_E)$ be an infinitesimal local symmetry of the Lagrangian system (L, Ω). Then the differential operator $\mathcal{D}(s)$ vanishes on all sections of $\Gamma(\pi)$.*

A more sophisticated version of the Second Noether Theorem arises if one requires that the bundle (A, p, M) is a bundle of Lie algebras. In this case $\Gamma(p)$ naturally inherits a similar algebraic structure, and $\mathcal{P}$ is required to be a morphism of Lie algebras. This implies some closure properties on the local coefficients of $\mathcal{P}$, but does not alter the generalized Bianchi identities.

Now we want to consider the relation between local symmetries and non-regularity (or degeneration) of the Lagrangian system. The notion of regularity arises from a choice of the Hamiltonian formalism and is expressed by properties of a conveniently defined Legendre transformation, which ensures the equivalence between the Lagrangian formalism and the Hamiltonian one. In the various approaches to a "covariant" Hamiltonian formalism (as opposed to a *space–time split* formalism), different phase-bundles are proposed in the literature, whether carrying a "multisymplectic structure" (see e.g. [10,4,26]), or a vector-valued symplectic form on each fiber [18,15]. However, at least for what concerns first-order theories, the definitions of regularity proposed in these works amount to say that the map $\lambda(L)$, defined in (3.1), which we will call for brevity *covariant Legendre transformation*, is a submersion at each point of J.

As shown in [9], regularity implies that the Euler–Lagrange equations are equivalent to the De Donder–Cartan equations, whose solutions are the sections $\rho \in \Gamma_U(J_M)$ satisfying

$$\rho^* (\xi \rfloor \mathrm{d}\mathbf{C}_L(\Omega)) = 0, \quad \forall \xi \in \Gamma(V J_M). \tag{4.4}$$

A second "Hamiltonian approach" (and the most usual for Field Theorists) passes through a decomposition of the space–time manifold into space- and time-component, the construction of a Lagrangian function on the tangent bundle of an infinite-dimensional manifold $T\mathcal{Q}$; after that one is in a position to "mimic" what is commonly done for a mechanical system, where one can pass to the cotangent bundle $T^*\mathcal{Q}$ by means of a suitably defined Legendre transformation. Since the cotangent bundle of an infinite-dimensional manifold is a problematic notion (unless $\mathcal{Q}$ is a Hilbert or Banach manifold), we will limit our considerations to the Lagrangian framework only. "Regularity" here simply implies the equivalence between the Euler–Lagrange equations and the *equations of motion* for the given Lagrangian on $T\mathcal{Q}$: see below Eq. (5.3) and also [23,1]. The relevance of this equation in the treatment of Lagrangian constraints for a mechanical system is shown in [20] and [12,13].

5. Degeneration in the instantaneous formalism

Let us interpret M as the space–time manifold. A "physical observer" can be represented by a *congruence* of time-like lines, which define a local splitting of M. Such a splitting can be performed in several ways (see e.g. [11]). Here, we assume the existence of a local diffeomorphism $\varphi : I \times \Sigma \longrightarrow U$, where U is an open subset of M and I is an open interval of $\mathbb{R}$ containing 0. Let us identify $\varphi(0, \Sigma)$ with Σ for the sake of simplicity. It is clear that in a relativistic framework Σ must be considered as a *space-like* surface, and the local curves $\varphi(\cdot, \mathbf{x}) : I \longrightarrow U$, $\mathbf{x} \in \Sigma$, as time-like curves, or rather as the world-lines of point-particles at rest in the frame of reference of the observer.

The tangent bundle TM splits *over* U: $T_m M \simeq T_t I \oplus T_{\mathbf{x}} \Sigma$, where $\varphi(t, \mathbf{x}) = m$, $m \in U$. Let us define $\Phi(m) = (\mathrm{d}/\mathrm{d}\tau)_{\tau=t}\, \varphi(\tau, \mathbf{x})$, with $m = \varphi(t, \mathbf{x})$. Moreover, for the sake of simplicity, we will identify M with U from now on.

The physically relevant situation is the one in which the values of the fields and their time derivatives are "smoothly" given at all (space-) points of Σ, these values being interpreted as the initial data for the dynamics. Thus, we must deal with sections *over* Σ, whose spatial derivatives are implicitly assumed to be known.

With this splitting of the space–time manifold, we can "distinguish" between spatial and temporal derivatives of the fields. This is formalized as follows. In the language of jet-bundles, we define the sub-bundles $J^{\mathcal{T}}$, $J^{\mathcal{S}}$ of J in the following way. For a given local section s of (E, π, M), let us denote, respectively, by $T_m s_{|I}$, $T_m s_{|\Sigma}$ the restrictions of the tangent map $T_m s$ to $T_t I$ and $T_x \Sigma$.

Definition. $J^{\mathcal{T}}$ is the disjoint union of the fibers $\left(J^{\mathcal{T}}\right)_m$, where each fiber consists of the following equivalence classes of local sections of (E, π, M):

$$\left(s \sim s'\right)^{\mathcal{T}}_m \quad \text{if } T_m s_{|I} = T_m s'_{|I}.$$

In the same fashion, $J^{\mathcal{S}}$ is the disjoint union of the fibers $\left(J^{\mathcal{S}}\right)_m$, where each fiber consists of the following equivalence classes of local sections of (E, π, M):

$$\left(s \sim s'\right)^{\mathcal{S}}_m \quad \text{if } T_m s_{|\Sigma} = T_m s'_{|\Sigma}.$$

Both $J^{\mathcal{S}}$ and $J^{\mathcal{T}}$ are affine bundles over E, whose associated vector bundles are $VE \otimes_E T^*\Sigma$ and $VE \otimes_E T^*I \simeq VE$ respectively. J is isomorphic in a natural way to the direct sum $J^{\mathcal{T}} \oplus_E J^{\mathcal{S}}$ (the sum is a fiberwise sum of affine spaces): the jet bundle decomposes into its space- and time-components. In a similar manner, the jet-extension operator $j : \Gamma(\pi) \longrightarrow \Gamma(\pi_1)$ decomposes as $j = j^{\mathcal{T}} \oplus j^{\mathcal{S}}$.

Local coordinates on $J^{\mathcal{T}}$ and $J^{\mathcal{S}}$ are given by, respectively,

$$\left(x^\mu, y^a, y^a_0\right), \qquad \left(x^\mu, y^a, y^a_k\right), \quad k = 1, \ldots, m.$$

We are thus able to define the concept of "initial data" for the classical field theory: the set of the *initial data* is a section of $J^{\mathcal{T}}{}_{|\Sigma}$, which means the restriction to Σ of the bundle $J^{\mathcal{T}}$.

Once the initial data γ is given, one can reconstruct a section $\hat{\gamma}$ of the "total" jet-bundle J on Σ in the following way: let $\pi^{\mathcal{T}}_{1,0}$ be the projection of $J^{\mathcal{T}}$ onto E, and $s = \pi^{\mathcal{T}}_{1,0}(\gamma)$; then

$$\hat{\gamma} = \gamma \oplus j^{\mathcal{S}} s. \tag{5.1}$$

Locally, one has

$$\hat{\gamma}(x) = \left(\gamma^a(x), \gamma^a_0(x), \frac{\partial}{\partial x^k}\gamma^a(x)\right).$$

Let us suppose that Σ is compact: the spaces $\Gamma\left(J^{\mathcal{T}}_{|\Sigma}\right)$ and $\Gamma\left(E_{|\Sigma}\right)$ of the sections of the restriction to Σ of the bundles $\left(J^{\mathcal{T}}, \pi^{\mathcal{T}}, M\right)$ and (E, π, M), respectively, become Fréchét manifolds (see e.g. [22]). Moreover, the projection $\Pi : \Gamma\left(J^{\mathcal{T}}_{|\Sigma}\right) \longrightarrow \Gamma\left(E_{|\Sigma}\right) : \gamma \longmapsto s = \pi^{\mathcal{T}}_{1,0}(\gamma)$ is a projection of an affine bundle; indeed, the fiber $\Pi^{-1}(s)$ is an affine space consisting of the sections of the affine bundle $\hat{s}^* J^{\mathcal{T}}$.

By various means it can be shown that $\Gamma\left(J^{\mathcal{T}}_{|\Sigma}\right)$ is isomorphic to the tangent bundle $T\Gamma\left(E_{|\Sigma}\right)$, whose fiber $T_s\Gamma\left(E_{|\Sigma}\right)$ is the space of sections of s^*VE. For example, in [11], this is achieved assuming the existence of a vector field ζ on E, projectable on M, and always transversal to $\pi^{-1}(\Sigma)$. Another way of doing this is by means of a connection $\mathcal{C} : TM \times E \longrightarrow TE$, which can be used to lift on E the vector field Φ on M. Then, given a section $\gamma \in \Gamma\left(J^{\mathcal{T}}_{|\Sigma}\right)$ the corresponding element in the tangent space $T_s\Gamma\left(E_{|\Sigma}\right)$ is the map $\Sigma \longrightarrow s^*VE : \mathbf{x} \longmapsto T_{\mathbf{x}}\tilde{s} \cdot \Phi(\mathbf{x}) - \mathcal{C}(\Phi(\mathbf{x}), \tilde{s}(\mathbf{x}))$, with $\tilde{s}$ any local section belonging to the class $\gamma(\mathbf{x})$. We will assume that this identification has been achieved. Elements of $T\Gamma\left(E_{\Sigma}\right)$ will be denoted by γ as well. We will denote by Q the configuration manifold $\Gamma\left(E_{|\Sigma}\right)$.

This identification is the key for writing the instantaneous dynamics. Let us define an *instantaneous Lagrangian* on the space of initial data:

$$\mathcal{L}_\Sigma : TQ \longrightarrow \mathbb{R}.$$

Regarding Ω as a map $J \longrightarrow \bigwedge^m T^*M$ along π_1, one can define the inclusion $(\Phi \rfloor \Omega)(e) = \Phi \rfloor (\Omega(e))$ which gives an (m)-horizontal form (on J). Then we set

$$\mathcal{L}_\Sigma(\gamma) = \int_\Sigma \hat{\gamma}^*(L\,\Phi \rfloor \Omega).$$

Now we can pursue an approach which parallels the formalism of classical point-particles. Given a Lagrangian function on a tangent manifold TQ of a finite-dimensional manifold, one can define *intrinsically* (i.e. without any reference to the symplectic cotangent structure) the Lagrangian 1-form $\theta_L = (\partial L/\partial \dot{q}^a)\mathrm{d}q^a$ and the energy function $E_L = (\partial L/\partial \dot{q}^a)\dot{q}^a - L$, and give the equations of motion

$$-X \rfloor \mathrm{d}\theta_L = \mathrm{d}E_L, \tag{5.2}$$

whose solution is the dynamical vector field X on TQ. It is well known that, in the case in which $\mathrm{d}\theta_L$ is a symplectic form, the solution of (5.2) is a second-order vector field whose integral curves are solutions of the Euler–Lagrange equations. Then one can say that in this case L is regular if $\mathrm{d}\theta_L$ is non-degenerate (see e.g. [21]).

In the case of $\mathcal{L}_\Sigma$ on TQ, the Lagrangian 1-form is given by

$$\Theta_\Sigma : T(TQ) \longrightarrow \mathbb{R}, \qquad \Theta_\Sigma(X_\gamma) = \int_\Sigma \hat{\gamma}^* \left(X_\gamma \rfloor \mathbf{C}_L(\Omega)\right),$$

where X_γ – a tangent vector in γ – can be considered as a map $X_\gamma : \Sigma \longrightarrow \hat{\gamma}^*(VJ_\Sigma)$. If we write locally, adopting a rather non-standard but useful notation so that $\partial/\partial\gamma^a$ (respectively: $\partial/\partial\gamma_0^a$, $\mathrm{d}\gamma^a$, $\mathrm{d}\gamma_0^a$) means "the local field $\partial/\partial y^a(\partial/\partial y_0^a, \mathrm{d}y^a, \mathrm{d}y_0^a)$ evaluated along the section γ",

$$X_\gamma(\mathbf{x}) = X_\gamma^a(\mathbf{x})\frac{\partial}{\partial\gamma^a} + X_{0\gamma}^a(\mathbf{x})\frac{\partial}{\partial\gamma_0^a},$$

then we can write

$$\Theta_\Sigma(X_\gamma) = \int_\Sigma \left(\frac{\partial L}{\partial y_0^a}\right)_{\hat{\gamma}} X^a \Omega_0,$$

where the subscript $\hat{\gamma}$ means the composition of the function within parentheses with the section. We will write

$$\Theta_\Sigma(\gamma) = \int_\Sigma \left(\frac{\partial L}{\partial y_0^a}\right)_{\hat{\gamma}} \mathrm{d}\gamma^a \otimes \Omega_0.$$

Then, one can define the *energy function*. Let us consider the total Legendre transformation $\mathbf{Leg}_L(\Omega)$ as an application $J \longrightarrow V^*E \otimes_J \bigwedge^m T^*M$. Then, as previously, let us define

$$\left(\Phi \rfloor \mathbf{Leg}_L(\Omega)\right)(e) = \Phi \rfloor \left(\mathbf{Leg}_L(\Omega)(e)\right).$$

We can define

$$\mathcal{E}_\Sigma : TQ \longrightarrow \mathbb{R},$$
$$\mathcal{E}_\Sigma(\gamma) = -\int_\Sigma \hat{\gamma}^* \left(\Phi \rfloor \left(\mathbf{Leg}_L(\Omega) + L\Omega\right)\right) = \int_\Sigma \hat{\gamma}^* \left(\Phi \rfloor \mathbf{C}_L(\Omega)\right).$$

Locally,

$$\mathcal{E}_\Sigma(\gamma) = \int_\Sigma \hat{\gamma}^* \left(\frac{\partial L}{\partial y_0^a} y_0^a - L\right) \Omega_0.$$

Let us define $\Omega_\Sigma = -\mathrm{d}\Theta_\Sigma$. For the Lagrangian function $\mathcal{L}_\Sigma$, the equation of motion is

$$X \rfloor \Omega_\Sigma - \mathrm{d}\mathcal{E}_\Sigma = 0. \tag{5.3}$$

We will say that the Lagrangian is *regular* provided that Ω_Σ is a non-degenerate two-form. We remark that in this case, at least enlarging the infinite-dimensional manifold $\mathcal{Q}$ we are considering, so that it can be structured as a Banach manifold, the field X is a *second-order equation*, and we are granted that for any initial condition there is a solution for it, and its integral curves are solutions of the Euler–Lagrange equations for $\mathcal{L}_\Sigma$ (see [1]).

Now let us come to the question of degeneration of the instantaneous formalism in relation to the presence of local symmetries. The degeneration of Ω_Σ depends, roughly speaking, on the degeneration of $\partial^2 L/\partial y_0^b \partial y_0^a$. Indeed, let $X_\gamma,\ Y_\gamma \in T_\gamma(T\mathcal{Q})$. Then X_γ belongs to $\ker \Omega_\Sigma$ at γ if $\Omega_\Sigma\left(X_\gamma, Y_\gamma\right) = 0, \forall Y_\gamma$. Since locally,

$$\Omega_\Sigma\left(X_\gamma, Y_\gamma\right) = \int_\Sigma \left(\frac{\partial^2 L}{\partial y^b \partial y_0^a}\left(X^a Y^b - Y^a X^b\right) + \frac{\partial^2 L}{\partial y_k^b \partial y_0^a}\left(X^a \frac{\partial Y^b}{\partial x^k} - Y^a \frac{\partial X^b}{\partial x^k}\right) \right. \\ \left. + \frac{\partial^2 L}{\partial y_0^b \partial y_0^a}\left(X^a Y_0^b - Y^a X_0^b\right)\right)_{\hat\gamma} \Omega_0,$$

the arbitrariness of Y_γ leads to

$$\left(\frac{\partial^2 L}{\partial y_0^b \partial y_0^a}\right)_{\hat\gamma(\mathbf{x})} X^a(\mathbf{x}) = 0,$$

$$\left(\frac{\partial^2 L}{\partial y^b \partial y_0^a} - \frac{\partial^2 L}{\partial y^a \partial y_0^b} - \frac{\partial^3 L}{\partial x^k \partial y_k^a \partial y_0^b}\right)_{\hat\gamma(\mathbf{x})} X^b(\mathbf{x}) \\ - \left(\frac{\partial^2 L}{\partial y_k^a \partial y_0^b}\right)_{\hat\gamma(\mathbf{x})} \frac{\partial X^b}{\partial x^k}(\mathbf{x}) - \left(\frac{\partial^2 L}{\partial y_0^b \partial y_0^a}\right)_{\hat\gamma(\mathbf{x})} X_0^b(\mathbf{x}) = 0.$$

From this one can easily see that defining $\mathcal{O}$ to be the set

$$\mathcal{O} = \left\{ \mathbf{x} \in M \text{ such that } \det\left(\frac{\partial^2 L}{\partial y_0^b \partial y_0^a}\right)_{\hat\gamma(\mathbf{x})} = 0 \right\},$$

then Ω_Σ is non-degenerate in γ if and only if the measure of $\mathcal{O}$ is zero. We remark that the condition that the determinant of $\left(\partial^2 L/\partial y_0^b \partial y_0^a\right)$ be different from zero on J would give the equivalence between Euler–Lagrange and De Donder–Cartan equations *for spatially holonomic sections*, i.e.

$$\hat\gamma\left(\xi \rfloor \mathbf{C}_L(\Omega)\right) = 0, \quad \forall \xi \in \Gamma(VJ_M),$$

where $\gamma \in \Gamma\left(J_M^T\right)$.

Let us suppose now that (L, Ω) possesses an infinitesimal local symmetry. Then Eq. (4.3) holds, so the symbol of $\mathcal{D}$ must be identically zero. $\mathcal{D}$ is non-linear, so we must consider its linearization (see Appendix A); writing down the coordinate expression for the symbol of the operator $\mathcal{D}$ we can immediately deduce that its vanishing implies the degeneration of the temporal Hessian $\partial^2 L/\partial y_0^b \partial y_0^a$ of the Lagrangian. Indeed, the symbol of $\mathcal{D}$ at s is a map

$$\mathcal{S}(\mathcal{D}, s) : T^*M \longrightarrow s^*(VE) \otimes (\mathcal{D}(s))^* \left(VA^* \otimes \bigwedge^m T^*M \right), \tag{5.4}$$

whose explicit form is

$$\begin{aligned} &\mathcal{S}(\mathcal{D}, s) \cdot \alpha_\rho \, \mathrm{d}x^\rho \\ &= \sum_{\mu,\nu,|N|=k} \alpha^{N+\mu+\nu} B^b_{N\,i}(s(x)) \left(\frac{\partial^2 L}{\partial y^a_\mu \partial y^b_\nu} \right)_{js} \mathrm{d}_\nu y^a \otimes \left(\epsilon^i \otimes \Omega \right), \end{aligned} \tag{5.5}$$

where

$$B^b_{Ni}(s(x)) = f^b_{Ni}(s(x)) - \frac{\partial s^b}{\partial x^\rho} e^\rho_{Ni}(x).$$

Now we have a way of "selecting" the partial Hessian matrix $\partial^2 L/\partial y_0^b \partial y_0^a$ by means of the 1-form in T^*M: it is enough to restrict to T^*I, and still the symbol must vanish:

$$\begin{aligned} 0 &= \mathcal{S}(\mathcal{D}, s) \cdot \alpha_0 \, \mathrm{d}x^0 \\ &= \alpha_0^{k+2} B^b_{(0,\dots,0)i}(s(x)) \left(\frac{\partial^2 L}{\partial y^b_0 \partial y^a_0} \right) (js(x)) \, \mathrm{d}_\nu y^a \otimes \left(\epsilon^i \otimes \Omega \right), \quad \forall \alpha_0. \end{aligned}$$

This implies that the vector $B^b_{(0,\dots,0)i}(s(x))$ lies in the kernel of the reduced Hessian matrix, and hence, for sufficiently general symmetries, we have degeneration of the 2-form Ω_Σ.

6. Degeneration in the covariant formalism

Regarding the degeneration in the covariant formalism – which is linked to the surjectivity of $\lambda(L)$ as was already pointed out – let us observe that the tangent map of $\lambda(L)$ reads in local coordinates

$$\begin{pmatrix} \mathbf{1}_m & 0 & 0 \\ 0 & \mathbf{1}_e & 0 \\ \left(\dfrac{\partial^2 L}{\partial x^\mu \partial y^a_\mu} \right) & \left(\dfrac{\partial^2 L}{\partial y^b \partial y^a_\mu} \right) & \left(\dfrac{\partial^2 L}{\partial y^b_\nu \partial y^a_\mu} \right) \end{pmatrix}.$$

In general, there is no relation between degeneration in the instantaneous formalism and degeneration in the covariant one. In any case, let us work out conditions enabling us to say that a Lagrangian system with an infinitesimal local symmetry $\mathcal{P}$ is degenerate. Using (3.2) one can easily see that a vertical field ξ on E is a symmetry of (L, Ω) if and only if $\mathbf{L}_{\xi^{(1)}} L\Omega = \mathrm{d}\alpha$, where α is a totally horizontal m-form. We will say that $\mathcal{P}$ defines a *vertical infinitesimal local symmetry* if for every $\sigma \in \Gamma(p)$ the field $\mathcal{P}\sigma$ is vertical and $\mathbf{L}_{\mathcal{P}^{(1)}\sigma} L = 0$.

We remark that the case of Lagrangians with vertical infinitesimal local symmetries includes the relevant cases of the electromagnetic field and, more generally, of the Yang–Mills fields.

We can prove the following statement.

Theorem. *Let $\mathcal{P}$ define a vertical infinitesimal local symmetry for (L, Ω). Then the Lagrangian system is degenerate.*

Proof. From $\mathbf{L}_{\mathcal{P}^{(1)}\sigma} L = 0, \forall \sigma \in \Gamma(p)$ one has

$$\mathrm{d}_{J_{|E}} \mathbf{L}_{\mathcal{P}^{(1)}\sigma} L = 0, \quad \forall \sigma \in \Gamma(p).$$

In local coordinates the previous identity reads

$$0 = \left(\frac{\partial^2 L}{\partial y^b_\nu \partial y^a} \xi^a + \frac{\partial^2 L}{\partial y^b_\nu \partial y^a_\mu} \frac{\mathrm{d}\xi^a}{\mathrm{d}x^\mu} + \frac{\partial L}{\partial y^a_\nu} \frac{\partial \xi^a}{\partial y^b} \right)_{js} \mathrm{d}y^b_\nu,$$

where locally we have set

$$\mathcal{P}\sigma = \xi^a \, \partial / \partial y^a .$$

Then for any section $s \in \Gamma(E)$ the following linear differential operator $\mathcal{O}_s$ has to vanish:

$$\mathcal{O}_s : \Gamma(p) \longrightarrow \Gamma\left(js^*\left(V^* J_E\right)\right), \qquad \mathcal{O}_s = \left(\mathrm{d}_{J_{|E}} \mathbf{L}_{\mathcal{P}^{(1)}\sigma} L\right)(js)$$

and the same must happen to its symbol:

$$\mathcal{S}(\mathcal{O}_s) : T^*M \longrightarrow \left(A^* \otimes_M js^*\left(V^* J_E\right)\right).$$

Locally, this is given by

$$\mathcal{S}(\mathcal{O}_s)\left(\alpha_\mu \mathrm{d}x^\mu\right) = \sum_{a,b,i,\nu,\mu,|N|=k} \alpha^{N+\mu} \left(f^a_{Ni} \frac{\partial^2 L}{\partial y^b_\nu \partial y^a_\mu} \right)_{js} \mathrm{d}y^b_\nu \otimes \epsilon^i = 0,$$

Then, for the arbitrariness of s one has

$$\sum_{\mu,a} \left(\sum_{|N|=k} \alpha^{N+\mu} f^a_{N,i} \right) \frac{\partial^2 L}{\partial y^b_\nu \partial y^a_\mu} = 0, \quad \forall b, \nu, i,$$

and this clearly shows the non-regularity of the Lagrangian system. □

Hence we can conclude that *if a Lagrangian system has vertical infinitesimal local symmetries, then it is degenerate both in the covariant and the instantaneous formalism.*

However, we may ask whether this is a complete characterization of the link between vertical local symmetries and degeneration of covariant systems. Indeed, the following counterexample shows that we can have a Lagrangian system which is degenerate both in the covariant framework and in the instantaneous one, but which does not seem to admit a vertical local symmetry.

Consider the Proca Lagrangian (the Lagrangian of the electromagnetic field with a "massive photon": A is a 1-form on $\mathbb{R}^4$, and $F = \mathrm{d}A$)

$$L_m = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}m^2 A_\mu A^\mu .$$

We can consider L_m as being defined locally on the trivial vector bundle $\mathbb{R}^4 \times \mathbb{R}^4 \longrightarrow \mathbb{R}^4$. It is straightforward to see that L_m is degenerate both in the covariant and in the instantaneous framework (since the only part of it which contributes to the Hessian is the "free-field" part). Furthermore, it is known [8] that the constraints algorithm yields two second-class ones for L_m. Then we argue that L_m does not admit any vertical infinitesimal local symmetry (as one might already guess by observing that L_m is the gauge-fixed Lagrangian of the electromagnetic field). We wish to state this in a more rigorous manner.

Suppose $E = \mathbb{R}^4 \times \mathbb{R}^4$, $M \subset \mathbb{R}^4$, and that $L : J \longrightarrow \mathbb{R}$ admits such a symmetry, and let us denote it as usual with $\mathcal{P}^{(1)}\sigma$. Suppose $I \times \Sigma \subset \mathbb{R} \times \mathbb{R}^3$, with I a finite interval and Σ a compact (spacelike) submanifold. We can identify smooth maps $s : I \times \Sigma \longrightarrow \mathbb{R}^4$ with smooth curves of sections $t \longmapsto s_t$ in $\mathcal{Q} = C^\infty(\Sigma, E)$ (see [22]). Then we see that

$$\begin{aligned}
0 &= \int\limits_{I\times\Sigma} js^*\left(\mathcal{P}^{(1)}\sigma \rfloor \mathrm{d}L\right)\Omega \\
&= \sum_{|N|\le k}\int\limits_I \mathrm{d}t \int\limits_\Sigma \Omega_0\Bigg(\frac{\partial L}{\partial y^a}\left(D^N\sigma^i(t,\mathbf{x})\right) f^a_{Ni}\left(t,\mathbf{x},s^b(t,\mathbf{x})\right) \\
&\qquad + \frac{\partial L}{\partial y^a_\mu}\frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\left(D^N\sigma^i(t,\mathbf{x})\right) f^a_{Ni}\left(t,\mathbf{x},s^b(t,\mathbf{x})\right)\right)\Bigg) \\
&= \int\limits_I \mathrm{d}t\, X(t)\rfloor \mathrm{d}\mathcal{L}_\Sigma(\dot{s}_t),
\end{aligned}$$

where $\mathcal{L}_\Sigma$ is defined in the usual way on $T\mathcal{Q} = C^\infty(\Sigma, E) \times C^\infty(\Sigma, E)$, and $X(t)$ is a time-dependent vector field on $T\mathcal{Q}$ defined by

$$\begin{aligned}
&X(t)(c,\dot{c}) : \Sigma \longrightarrow E \times E, \\
&X(t)(c,\dot{c})(\mathbf{x}) = \sum_{|N|\le k}\Bigg(D^N\sigma^i(t,\mathbf{x})\, f^a_{Ni}(t,\mathbf{x},c(\mathbf{x}))\frac{\partial}{\partial c^a} \\
&\qquad + \Bigg(\left(\frac{\mathrm{d}}{\mathrm{d}x^0}D^N\sigma^i(t,\mathbf{x})\right) f^a_{Ni}(t,\mathbf{x},c(\mathbf{x})) \\
&\qquad + D^N\sigma^i(t,\mathbf{x})\left(\frac{\partial f^a_{Ni}}{\partial x^0} + \frac{\partial f^a_{Ni}}{\partial y^b}\dot{c}^b(\mathbf{x})\right)\Bigg)\frac{\partial}{\partial \dot{c}^a}\Bigg).
\end{aligned}$$

Furthermore, $X(t)\rfloor \mathrm{d}\mathcal{L}_\Sigma(\dot{s}_t) = (\mathrm{d}F/\mathrm{d}t)_{s_t}$. The relations between local symmetries and constraints is developed in the instantaneous formalism often disregarding the problem of infinite dimensionality. In this context, the analysis of the equivalence between gauge transformations (i.e. symmetries depending upon arbitrary functions of time) and first-class constraints dates back to Dirac's works in a Hamiltonian setting. More recently, it has been pointed out that not all Lagrangian gauge symmetries are projectable on Hamiltonian gauge symmetries (see [20,14]). It is easy to see that in our case $X(t)$ is the lift to TQ of the vector field ξ on Q,

$$\xi(t)(c)(\mathbf{x}) = \sum_{|N|\leq k} D^N \sigma^i(t,\mathbf{x})\, f^a_{Ni}(t,\mathbf{x},c(\mathbf{x}))\, \frac{\partial}{\partial c^a}.$$

This remark implies that $X(t)$ is a *projectable* local symmetry for $\mathcal{L}_\Sigma$. We can thus apply the results of [2], which show that such a Lagrangian must generate (primary) first-class constraints.

So we conclude that, in the framework we have outlined, there exist Lagrangians which are degenerate both at the covariant and at the instantaneous level, but which do not admit vertical infinitesimal local symmetries.

We must point out that the Lagrangian L_m is indeed the prototype of a general category of Lagrangians: every time one "fixes the gauge", the first-class constraints are automatically turned into second-class ones. Finally, we remark that a slightly different concept of "symmetry" might lead to an "extended Second Noether Theorem" (see [19]), and this in turn might alter the conclusions we have drawn.

7. The case of higher-order Lagrangians

In the last sections we showed how degeneration can appear in the instantaneous and in the covariant formalism in the case of first-order Lagrangians which possess a local symmetry.

Here we sketch how these considerations can be adapted to the case of higher-order Lagrangians with local symmetries. However, in that case the transition to the instantaneous and to the covariant formalism is a more problematic notion. We refer to the two papers [10,11] for an extensive account on this topic. The main problem in the treatment of a higher-order Lagrangian $L : J^l \longrightarrow \mathbb{R}$ is that, for $m > 0$ and $l \geq 2$, many differential forms $\mathbf{C}_L(\Omega)$ – called Cartan forms – arise on J^{2l-1} which can play the role of the form defined in Section 3 for the first-order case. They are $(m+1)$-forms on J^{2l-1} satisfying the following relations:

$$\begin{aligned}
&\mathbf{C}_L(\Omega) - \pi^*_{2l-1,l}(L\Omega)(X_0,\ldots,X_m) = 0, \qquad \text{when } X_0,\ldots,X_m \in \mathcal{H}_{2l-1},\\
&(X \rfloor \mathrm{d}\mathbf{C}_L(\Omega))(Y_0,\ldots,Y_m) = 0, \quad \forall X \in \mathcal{V}_{2l-1},\ \forall Y_0,\ldots,Y_m \in \mathcal{H}_{2l-1},\\
&X \rfloor \mathbf{C}_L(\Omega) = 0, \quad \forall X \in VJ^{2l-1}_{|J^{l-1}},\\
&X \rfloor Y \rfloor \mathbf{C}_L(\Omega) = 0, \quad \forall X, Y \in VJ^{2l-1}_{|E}.
\end{aligned}$$

The set of solutions of the first-order equation analogous to the De Donder–Cartan equation (4.4) contains the solutions of the Euler–Lagrange equation, but even in the *best situation* a solution ρ of the De Donder–Cartan equation, which is a section of J^{2l-1}, is in general not holonomic, but only projects down to a *holonomic section of* J^l. However, it is possible to show that the transition to the instantaneous formalism is not ambiguous. Indeed, once that the space+time decomposition is performed, one can define on the manifold of initial data $J^{\mathcal{T}}_{\Sigma} \subset J^{2l-1}_{\Sigma}$ (defined in a way analogous to the first-order case), the Lagrange 1-form Θ_{Σ} and the energy function $\mathcal{E}_{\Sigma}$ as in Section 5. These objects do not depend on the particular choice of the Cartan form and lead to a well-defined equation of motion analogous to (5.3) (see [11]).

If one defines a local symmetry, in the case of higher-order Lagrangians, simply extending the definition of Section 2 to the lth-order, one still obtains "Bianchi identities" of the form

$$0 = \mathcal{D}(s) = {}^{t}\mathcal{P}^{(2l)}_{s} \cdot \mathbf{E}_{L}\,(\Omega)\,(j^{2l}s), \quad \forall s \in \Gamma(E),$$

and using the space+time splitting to select the temporal coordinates, one obtains the degeneration of the "temporal Hessian" $\partial^{2}L/\partial y^{b}_{0_l}\partial y^{a}_{0_l}$ (here 0_l means a l-multi-index made up of zeros). This implies the degeneration of the two-form $\Omega_{\Sigma} = -\mathrm{d}\Theta_{\Sigma}$ which appears in the equation of motion (5.3). Indeed, choosing a vector

$$X_{\gamma} = X^{a}(\gamma)\frac{\partial}{\partial y^{a}_{0_{2l-1}}} \in T_{\gamma}J^{\mathcal{T}}_{\Sigma}, \quad \text{with } X^{a}\frac{\partial^{2}L}{\partial y^{b}_{0_l}\partial y^{a}_{0_l}} = 0,$$

it is easy to see that X_{γ} belongs to the kernel of Ω_{Σ}, since using the local expression given in [11] one sees that for any tangent vector Y_{γ}

$$\Omega_{\Sigma}\left(X_{\gamma}, Y_{\gamma}\right) = \int_{\Sigma} \frac{\partial^{2}L}{\partial y^{b}_{0_l}\partial y^{a}_{0_l}}\,\mathrm{d}y^{b}_{0_{2l-1}} \wedge \mathrm{d}y^{a}\left(X_{\gamma}, Y_{\gamma}\right) = 0.$$

On the other hand, the transition to the covariant Hamiltonian formalism depends on the choice of the Cartan form; each of these forms generate a different *covariant Legendre transformation*, and one has the best situation between the Euler–Lagrange and the De Donder–Cartan equations if the corresponding Legendre transformation has maximal rank at each point. This notion of regularity depends on the choice of the Cartan form unless one restricts to the *Poincaré–Cartan forms* (as shown in [10]), or otherwise one can make a particular choice of $\mathbf{C}_{L}(\Omega)$ as in [27]. In all these cases the condition $\det\left(\partial^{2}L/\partial y^{b}_{M}\partial y^{a}_{N}\right) = 0$ implies the non-regularity of the Lagrangian (M and N are multi-indices of rank l). If a Lagrangian system has a vertical infinitesimal local symmetry, i.e. $\mathbf{L}_{\mathcal{P}^{(l)}\sigma}L = 0$, $\forall\sigma$, then one can show along the same lines of Section 6 that this condition is satisfied. The only change needed is that one must evaluate the symbol of the differential operator $\mathrm{d}_{J^{l}_{|J^{l-1}}}\mathbf{L}_{\mathcal{P}^{(l)}\sigma}L$, where $\mathrm{d}_{J^{l}_{|J^{l-1}}}$ means the vertical derivative on the affine bundle $J^{l} \longrightarrow J^{l-1}$.

Appendix A. Differential operators

Let (F_i, p_i, M), $i = 1, 2$ be ordinary fiber bundles. A *differential operator of order l* is a map $\mathcal{O} : \Gamma(p_1) \longrightarrow \Gamma(p_2)$ given by $\mathcal{O}(s) = O_{*}j^{l}s$ for $s \in \Gamma(p_1)$, where $O : J^{l}(F_1) \longrightarrow F_2$ is a smooth bundle morphism. If (F_i, p_i, M), $i = 1, 2$ are vector bundles and O is a vector bundle morphism, then $\mathcal{O}$ is said to be a *linear differential operator*. If $\Gamma(p_i)$, $i = 1, 2$ are given the structure of Fréchét spaces as in [6], differential linear operators are identified with continuous local linear maps from $\Gamma(p_1)$ to $\Gamma(p_2)$ (where "local" means that for every section s such that $s(x) = 0, \forall x \in U$, $U \subset M$ open, then $\mathcal{O}(s(x)) = 0, \forall x \in U$).

Given a linear differential operator $\mathcal{O} : \Gamma(p_1) \longrightarrow \Gamma(p_2)$, one can define the *transposed operator* ${}^t\mathcal{O}$ as follows (cf. [9]): it is the only linear differential operator ${}^t\mathcal{O} : \Gamma(p_2^* \otimes \bigwedge^m T^*M) \longrightarrow \Gamma(p_1^* \otimes \bigwedge^m T^*M)$ such that $\forall \phi_2 \in \Gamma(p_2^* \otimes \bigwedge^m T^*M)$

$$\int_M \langle f_1, {}^t\mathcal{O}(\phi_2)\rangle = \int_M \langle \mathcal{O}(f_1), \phi_2\rangle, \quad \forall f_1 \in \Gamma_c(p_1).$$

Let $\mathcal{O}$ be a *linear* differential operator. Let $\alpha \in T_x^*M$, $a \in F_{1x}$; let $f : M \longrightarrow \mathbb{R}$ be such that $f(x) = 0$ and $\mathrm{d}_x f = \alpha$; let further $s \in \Gamma(p_1)$, $s(x) = a$. Then the *symbol* of $\mathcal{O}$ is the homomorphism

$$\mathcal{S} : T^*M \longrightarrow F_1^* \otimes F_2, \quad \mathcal{S}(\mathcal{O})(\alpha) \cdot a = \mathcal{O}\left(\frac{1}{l!} f^l s\right)(x) = \frac{1}{l!} O\left(j^l f^l s(x)\right).$$

If (F_i, p_i, M), $i = 1, 2$ are general fiber bundles, one can give to $\Gamma(p_i)$ the structure of an infinite-dimensional manifold, as in [22], whose tangent space $T_s\Gamma(p_i)$ in s is the space $\Gamma_c(s^*VF_i)$ of the sections of the vertical bundle VF_i "covering s" and with compact support, i.e. sections $v : M \longrightarrow VF_i$ such that $\tau_{F_i} \cdot v = s$.

A differential operator results in a differential map from $\Gamma(p_1)$ to $\Gamma(p_2)$ whose tangent map at s is $T_s\mathcal{O} : \Gamma_c(s^*VF_1) \longrightarrow \Gamma_c(\mathcal{O}s^*VF_2) : v \longmapsto VO \cdot j^l v$, where $VO : VF_1 \longrightarrow VF_2$ is the restriction of the tangent map TO to the vertical bundles. Thus, for every $s \in \Gamma(p_1)$, a linear differential operator $\Lambda(\mathcal{O})_s : \Gamma(s^*VF_1) \longrightarrow \Gamma(\mathcal{O}s^*VF_2)$ is given by extension. We define the *symbol* $\mathcal{S}(\mathcal{O}, s)$ of the non-linear operator $\mathcal{O}$ at s as the symbol of $\Lambda(\mathcal{O})_s$. If $\mathcal{O}_s$ is given locally by the maps $f_j\left(x, s^i(x), D^N s^i(x)\right)$, $i = 1, \ldots, r_1$, $j = 1, \ldots, r_2$, where r_i is the dimension of the fiber of p_i, then

$$\mathcal{S}(\mathcal{O}, s)_{ij} \cdot \alpha_x = \sum_{|N|=l} \alpha_x^N \frac{\partial f_j}{\partial y_i^N}\left(s^h(x), D^N s^h(x)\right),$$

where $\alpha_x \in T_x^*M$ and $\alpha_x^N = (\alpha_x)_1^{N_1} \cdots (\alpha_x)_m^{N_m}$. For the definition of the symbol for linear and non-linear operators, and also for its globalization, see [24].

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